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FINDING MINIMUM SPANNING TREES WITH A FIXED NUMBER OF LINKS AT A NODE

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FINDING MINIMUM SPANNING TREES WITH A FIXED NUMBER OF LINKS AT A NODE

by

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ABSTRACT

This paper addresses a variant of the minimum spanning tree problem in which a given node is required to have a fixed number of incident edges. We show that this problem, which is combinatorially a level of complexity beyond the ordinary minimum spanning tree problem, can be solved by a highly efficient "quasi-greedy" algorithm. Applications include a tele-communication linking problem and a new relaxation strategy for the traveling salesman problem via appropriately defined order-constrained one-trees.

1. Introduction

The minimum weight spanning tree problem has enjoyed a good deal of notoreity ever since Kruskal first provided a greedy algorithm for solving it [12]. Interest in the problem at least in the beginning, appeared to center primarily around the novelty that something with a nontrivial statement could be solved by an "almost trivial" procedure. Philosophically, this was both intriguing and unsettling, and other manifestations and generalizations of greedy algorithms were sought [1, 13]. A broad characterization of such methods in the context of matroid theory was accomplished by Edmonds [3], who coined the term "greedy algorithm."

With rare exception (e.g., [4]), the precise form assumed by a greedy algorithm is usually one of the first possibilities that springs to mind, and the validity of such an approach can typically be established without notable effort. The early applications seemed for some time to have little practical significance and little relevance outside their immediate contexts. Recently, however, things have changed. Practical applications in such diverse areas as least cost electrical wiring, minimum cost connecting communication and transportation networks and minimum stress networks have found their way into the literature and textbooks (see, e.g., [10a, 11, 15]). A variation of the minimum spanning tree problem, called the minimum "1-tree" problem was shown by Held and Karp [8,9] to be extremely useful as a relaxation of the traveling salesman problem. In addition, Dakin [1], Kershenbaum and Van Slyke [11] have shown that there is more to the implementation of greedy algorithms than previously suspected, and have developed rather ingenious procedures for organizing and updating the information used by a greedy algorithm to improve its efficiency

Throughout all this flurry of activity, an extremely important relative of

the minimum spanning tree problem has surprisingly been neglected: that of determining a minimum weight spanning tree subject to the additonal restriction that a given node be constrained to a specified order (i.e., have a fixed number of incident edges). Such a problem is directly relevant in the traveling salesman context, where nodes are constrained to order 2. The problem also arises, with perhaps greater practical immediacy, in a telecommunications setting. Here the objective is to find the minimum cost way of setting up transmission cables to connect users in various cities to a common computer installation. The "order constraint" derives from the requirement that the immediate links to the computer facility must be at least of a certain number, in order to accommodate the fact that too few links will be unable to support the anticipated transmission load. (The requirement that a node be constrained to "at least" or "at most" a certain order can be handled as a simple variant of constraining it to be exactly of that order.)

In view of the foregoing remarks, the purpose of this paper is to address the following problem: P(K) - <u>Find a minimum weight spanning tree with node 0 constrained to order K.</u> Here, as customary, we implicitly have reference to an underlying graph of nodes and edges, and the weight of a subgraph (hence a spanning tree) is defined to be the sum of the weights of the edges in that subgraph. Node 0 may of course represent any selected node in the graph, and K is assumed to be a positive number for which a spanning tree with exactly K edges incident to node 0 exists. (Otherwise, the solution of P(K) will determine the nonexistence of such a tree.)

Our principal results for characterizing optimal spanning trees with a constrained order at node 0 consist of a "primal theorem" and a "dual theorem". The former gives a method for constructing an optimal tree beginning with any tree that already satisfies the order requirement at node 0, and the latter gives a method for constructing an optimal tree of order K + 1, or K - 1 at node 0 (as desired) from an optimal tree with order K at node 0.

The dual theorem is in fact a characterization of a "quasi-greedy" algorithm, for it makes the very best move from the category available to it, not by "putting things into a bucket" (in Edmond's terminology) but by trading things between two buckets.

We also provide special labeling procedures that enable the primal and dual methods to be applied by means of "modified pivot steps" analogous to the basis exchange steps employed in specialized linear programming procedures for solving minimum cost flow network problems. The "modified pivot steps", of course, do not involve the use of a specialized linear programming algorithm, since the problem under consideration is combinatorial and has no LP network equivalent; however, the amount of calculation of these modified pivot steps is in fact on the same order as--or somewhat better than--that of an LP basis exchange in a network. Further in the dual case each step immediately gives an **optimal** spanning tree of the next higher or lower order at node 0, thereby producing an algorithm of considerable efficiency. In the concluding section we discuss how this "quasi-greedy" algorithm can be similarly applied to the constrained minimum one-tree problem enhancing the significance of this method for the traveling salesman problem.

2. Notation and Results

To lay the groundwork for the primal and dual theorems for constructing optimal ordered-constrained trees we introduce the following definitions and notational conventions. T and T' will denote distinct spanning trees, defined on a common graph. We also allow T and T' to represent the sets of edges for these trees, writing for example, $e \in T - T'$ to indicate that e is an edge in T but not in T'.

The unique edge-simple path in T connecting the endpoints of an edge e will be denoted T(e)(and likewise will interchangeably be used to represent the set of edges for this path). For two edges e, e' such that $e \in T$ and $e' \notin T$, we will

call the process of adding e' to T and deleting e from T an admissible exchange (relative to T) if the result is also a spanning tree. Thus, in particular, such an exchange is admissible if and only if $e \in T(e')$.

Using these definitions, we will first state a theorem of [7] concerning the existence of a special "matching" of edges from T and T' that is particularly useful for establishing the main results of this paper. (Due to its subordinate role in the present setting, we state it as a lemma.)

Lemma 1. For any two distinct spanning trees T and T', there is a way of pairing the edges of T - T' with those of T' - T (in a one-one matching) so that every pair gives an admissible exchange relative to T.

The proof of this result in [7] gives a constructive procedure for producing a pairing that satisfies the stated conditions. Such a construction will not concern us here, but we require an additional preliminary (and somewhat non-intuitive) result to complete the foundation for our principal theorems.

Lemma 2. Assume that e_0, $e \in T$, e_0', $e' \notin T$ and e_0 and e_0' are incident to the same node. Further assume that at least one of the pairs e, e' and e_0, e'_0 does not give an admissible exchange relative to T(by deleting the first member of the pair and adding the second). Then e_0, e' and e, e_0' both yield an admissible exchange relative to T if and only if the addition of e' and e_0' and the deletion of e_0 and e result in a spanning tree (hence, if and only if the pairs e_0, e' and e, e_0' yield successively admissible exchanges, executed in either order).

Proof: For the "only if" part, assume $e_0 \in T(e')$ and $e \in T(e_0')$. Swapping e' and e_0 gives a tree T' in which e_0' and e still give an admissible exchange unless $T'(e'_0) \neq T(e'_0)$, which occurs only if $e_0 \in T(e_0')$, implying e_0 can exchange admissibly with e_0'. By assumption it follows that $e \notin T(e')$ (else e and e' could exchange admissibly). Thus $e \in T(e_0') - T(e")$, and it

follows that the edge simple path $T(e') \cup \{e'\} \cup T(e_0') - (T(e') - T(e_0'))$ in fact contains e and is $T'(e_0')$. Thus the second exchange is admissible in T', proving that a tree results. (A similar argument leads to the same conclusion by considering the swaps in reverse order.) For the "if" part of the lemma, assume that $T \cup \{e_0', e'\} - \{e_0, e\}$ is a tree. By Lemma 1 there is some way of pairing the edges, e_0', e' with the edges e_0, e so that every pair gives an admissible exchange relative to T. If e_0' cannot be paired with e_0 or if e' cannot be paired with e, this leaves the two pairings, e, e_0' and e_0, e' by default. The equivalence of the statement that these two pairings give successively admissible swaps when executed in either order follows immediately from the foregoing.

For the statement of the following "primal" theorem, we call an admissible exchange <u>improving</u> if the resulting tree has a smaller weight than the original (hence if the weight of the added edge is less than the weight of the deleted edge). We also follow the convention that an edge is incident to node 0 if and only if it is subscripted with a "0".

<u>Theorem 1</u> (Primal Approach): A spanning tree T with order K at node 0 is optimal for problem P(K) if and only if

(1) There are no improving admissible exchanges involving a pair e, e', where $e \in T$, $e' \notin T$ (and neither edge is incident to node 0);

(2) There are no improving admissible exchanges involving a pair e_0, e_0', where $e_0 \in T$, $e_0' \notin T$(and both edges are incident to node 0);

(3) There are no two exchanges, both admissible relative to T, involving a pair e_0, e' and a pair e, e_0', such that $e_0, e \in T$, $e_0', e' \notin T$, which together yield a net improvement--i.e., for which the sum of the weights of e_0' and e' are less than the sum of the weights of e_0 and e. (In particular, this says that coupling the "best admissible pair" of the form e_0, e' with the best admissible pair" of the form e_0' , e does not yield a net improvement, disregarding

whether the exchanges can actually be carried out in sequence.)

<u>Proof:</u> First we prove the "only if" part of the theorem. Clearly if there are any improving exchanges of the type indicated in (1) or (2), then T is nonoptimal. If there are no such exchanges but there exists a pair of exchanges such as described in (3), then we may assume that either e_0, e_0' or e, e' cannot give an admissible exchange, else at least one would be improving, contrary to assumption. But then by Lemma 2 the two exchanges of (3) can in fact be carried out sequentially, again establishing that T is nonoptimal. To prove the "if" part of the theorem, suppose that (1), (2) and (3) hold, but that there exists a spanning tree T' which is feasible for P(K) and has a smaller weight than T. By Lemma 1 we can match the edges of T - T' with those of T' - T so that each pair gives an admissible exchange in T. Since node 0 has the same order in both T and T', it follows that these admissible exchanges consist exactly of the types indicated in (1) and (2) together with the two types of exchanges indicated in (3), where the number of each of these two latter types is equal. Since the weight of T' is less than that of T, and since no admissible e, e' and no admissible e_0, e_0' exchanges are improving, it follows that the sum of weights of all the admissible exchanges of the e, e_0' and the e_0,e' type must be negative (adding the weights of the edges in T' and subtracting those of the edges in T). But then the sum of the weights of <u>some</u> admissible e, e_0' exchange and <u>some</u> admissible e_0,e' exchange (in particular, the "best" of each type) must be negative, contrary to the assumptions of (3). The contradiction establishes the theorem.

By means of the foregoing theorem we can now state and prove the two forms of the "dual" theorem for order-constrained spanning trees (expressed as Theorem 2 and its corollary), which show how to obtain optimal solutions for P(K+1) and P(K-1) from an optimal solution for P(K).

Theorem 2 (Dual Approach-Increasing order).

Assume T is optimal for P(K) and T' is obtained from T by applying a single admissible exchange involving the edges e_0',e where $e \in T$, $e_0' \notin T$ (e_0' is incident to node 0 and e is not incident to node 0), and the weight of e_0' less the weight of e is minimum over all admissible exchanges of the specified type. Then T' is optimal for P(K+1).

Proof: We will show that T' satisfies the optimality conditions of Theorem 1. First, we show that (1) holds. We may restrict attention to admissible exchanges of the form e_1, e_1' in T' that were not available in T. Such an exchange yields a tree $T'' = T \cup \{e_0', e_1'\} - \{e, e_1\}$ and by Lemma 1 e_0', e_1 and e_1', e must both give admissible exchanges in T. But the first of these is no better than e_0', e and the second is a nonimproving move, and hence T" is not better than T'. Next we establish condition (2). The admissible exchange of the form indicated in (2) applied to T', gives a tree $T'' = T \cup \{e_0', e_0''\} - \{e, e_0\}$ where $e_0 \in T'$ hence $e_0 \in T$ (disregarding $e_0 = e_0'$ which reduces to a tree already known to be no better than T') and $e_0'' \notin T'$ hence $e_0'' \notin T$. By Lemma 1, and the fact that e_0, e_0'' cannot give an admissible exchange in T, both e_0', e_0 and e_0'', e must give admissible exchanges in T. But the first is nonimproving and the second no better than e_0', e, and hence T' again cannot be improved. Finally, we show that (3) holds. A double exchange involving e_0'', e_1 and e'_1, e_0^* which yields a net improvement must be capable of being executed in sequence, applying Lemma 2 and the fact that conditions (1) and (2) have been established for T'. Here e_1, $e_0^* \in T'$ hence $\in T$, and e_0'', $e_1' \notin T'$ hence $\notin T$, disregarding $e_0^* = e_0'$ and $e = e'_1$, both of which reduce to earlier cases.

Thus we have a tree $T'' = T \cup \{e_0', e_0'', e_1'\} - \{e, e_1, e_0^*\}$ where the latter set of edges is from T and the former is not. Applying Lemma 1, these two sets

of edges may be matched in some way so that all resulting pairs give admissible exchanges in T. We shall examine the relevant "possible" matchings to determine their implications for T". First the pairing $\{e_0{*},e_1'\},\{e_1,e_0''\},\{e_0',e\}$ is not possible, because if the first two pairs had been admissible (i.e., given rise to admissible exchanges) in T they would have implied the nonoptimality of T in the same way we assume they imply the nonoptimality of T'. Similarly $\{e_0{*},e_0''\},\{e_1,e_1'\},\{e_0',e\}$ is impossible, because at least one of the first two pairs must be improving by the assumed improvement of T" over T', and the admissibility of such a pair in T violates its presumed optimality. This leaves the following cases: $\{e_0'',e\},\{e_1,e_0'\},\{e_0{*},e_1'\}$; $\{e_0'',e\},\{e_1,e_1'\},\{e_0{*},e_0'\}$; $\{e_0{*},e_1\},\{e_1',e\},\{e_0'',e_0'\}$; $\{e_0'',e_1\},\{e_1',e\},\{e_0{*},e_0'\}$. All of these may be ruled out because in each case the last two pairs are nonimproving (due to the optimality of T) and the first pair gives a tree no better than T'. This contradicts the postulated improvement of T" over T' and completes the proof.

From this theorem we may infer the following "inverse" result.

Corollary (Dual Approach-decreasing order): Assume T is optimal for P(K) and T' is obtained from T by applying a single admissible exchange involving the edges e_0,e', where $e_0 \in T$, $e' \notin T$ (e_0 is incident to node 0 and e' is not), and the weight of e' less the weight of e_0 is minimum over all admissible exchanges of the specified type. Then T' is optimal for P(k-1).

Proof: The corollary follows by essentially the same reasoning used to establish Theorem 2.

In Theorem 2 and its corollary, the absence of an admissible exchange that increases of decreases the number of edges incident to node 0 of course implies the nonexistence of a spanning tree of the resulting order at this node. (This is a direct consequence of the stated results and the use of "infinite weight" edges to represent those not contained in the graph.)

We now show how to take advantage of these theorems in an efficient manner.

3. Labeling Procedures

The identification of an admissible exchange that is the "best" of all admissible exchanges in its category, which is required by both Theorem 2 and its corollary (and also, indirectly by Theorem 1) appears at first glance to involve the computation of "exchange values" over a potentially vast number of partial chains. We will show in this section how to apply labeling procedures (different, but comparably efficient, in each of the three cases) that succeed in generating all such relevant values, with an amount of computation essentially no greater than that of evaluating updated objective function coefficients for nonbasic variables in specialized linear programming approaches to ordinary network problems. (We refer here to "streamlined" basis evaluation procedures such as those of [5,6,14].) In addition, we show how to apply the foregoing primal and dual results iteratively by means of correspondingly refined updating steps that impose minimal amounts of recalculation (likewise, comparable in efficiency to the approaches of [5,6]).

In all of the labeling procedures, it is assumed that the current spanning tree T is recorded as an arborescence with Johnson's "triple label" scheme [10], with the root at node 0. As customary, a node r will be called an <u>immediate successor</u> of node q if there is an edge in T incident on nodes q and r and if the unique path in the arborescence from r to the root contains node q. A node r will be called a <u>successor</u> of node q if the unique path from r to the root contains node q.

<u>Labeling Rule For The Dual Approach</u> - decreasing order

1. Assign a label $t_q = r$ to each successor q of an immediate successor r of node 0. To each immediate successor node r of node 0 also assign a label of $t_r = r$. Assign node 0 a label of 0.

2. For each edge $(i,j) \notin T$ whose node labels are not the same and for which neither i nor j is node 0, set

$$\theta_{ij} = w_{ij} - \max(w_{0t_i}, w_{0t_j})$$

where w_{ph} denotes the weight on edge (p,h). Set $\theta_{ij} = \infty$ for all other edges.

3. To determine the e_0, e' exchange of the corollary to Theorem 2: let $\theta_{rs} = \min_{(i,j)} \theta_{ij}$. If θ_{rs} is finite, then edge (r,s) = e', and e_0 is the edge associated with $\max(w_{0t_i}, w_{0t_j})$ in step 2. If θ_{rs} is infinite, no spanning tree of the desired order exists.

The validity of the foregoing procedure follows from the corollary to Theorem 2 and the fact that e_0, e' gives an admissible exchange if and only if $e_0' \in T(e')$.

The above procedure is clearly quite easy to implement. Additionally, the labels used in the procedure can easily be updated. Specifically, suppose an optimal spanning tree T for P(K) is known and an optimal spanning tree T' for P(K-R) (where R < K) is desired. The the above procedure can be successively used without completely re-labeling the nodes for each intermediate spanning tree. This is readily accomplished by using the API method [5] to update the rooted tree pointers (predecessor, successor, and brother indexes) and using the following observations to update the node labels.

Deleting edge e_0 splits T into two disjoint trees. One of these trees (say T_0) contains the root (node 0) and the other tree (say T_1) does not. (Note that all the node labels of T_1 are the same.) The addition of edge e'= (r,s) reconnects these trees. When T_1 is re-attached to T_0 via edge (r,s) then all node labels of T_0 are still correct and all node labels of T_1 should be changed to r if $r \in T_0$ or s if $s \in T_0$.

<u>Labeling Rule For The Dual Approach</u> - increasing order

1. Assign a label $w_r = 0$ to each immediate successor r of node 0, and assign node 0 a label of 0. To each immediate successor r of a node t whose node label has been set, assign a label $w_r = \max (w_t, w_{tr})$, where w_{tr} is the weight on edge (t,r).

2. To determine the e, e_0' exchange of Theorem 2:

   For each edge $(0,j) \notin T$, set $\theta_{0j} = w_{0j} - w_j$ and let $\theta_{0q} = \min\limits_{(0,j)\notin T} \theta_{0j}$.

   Then edge $(0,q) = e_0'$ and e is the edge in T (e'_0) whose weight is equal to w_q. (If this edge is not unique pick any such edge that is not incident to node 0).

The validity of this procedure follows directly from Theorem 2. As in the case of decreasing node order, the procedure is quite easy to implement and the labels can be updated with minimal effort. In particular, suppose an optimal spanning tree T for P(K) is known and an optimal spanning tree T' for P(K+R) is desired. Then the node labels can be easily updated using the observations similar to those made earlier. Deleting edge e split the minimum spanning tree into two disjoint trees. As before, one of the trees (say T_0) contains the root (node 0) and the other tree (say T_1) does not. The addition of edge e'_0 re-connects these trees. When T_1 is re-attached to T_0 via edge $e'_0 = (0,q)$ then all node labels of T_0 are still correct and thus only tree node labels of T_1 need to be updated. The updating of the node labels in T_1 occurs by setting $w_q = 0$ and then assigning a node label w_r to each immediate successor r of each node $t \in T_1$ (whose node label has been set) equal to $w_r = \max (w_t, w_{tr})$.

The comments made with respect to implementing the decreasing node order procedure apply in the present setting. Further the above procedure

is even more efficient since only the edges incident to node 0 and not in T have to be evaluated.

Labeling Rules For The Primal Approach

The foregoing labeling procedures can be adapted and integrated to yield an efficient labeling procedure for Theorem 1. In this approach, two labels must be kept for each node. One label corresponds to the node label used in the Dual Approach-decreasing order procedure and the other label corresponds to the node label used in the Dual Approach-increasing order procedure. These labels are determined as follows.

1. Assign a label $t_r = r$ and label $w_r = 0$ to each immediate successor node r of node 0. Assign node 0 a label $t_0 = 0$ and a label $w_0 = 0$. Then to each immediate successor k of a node r whose node labels have been set, assign a label $t_k = r(=t_r)$ and a label $w_k = \max(w_r, w_{kr})$, where w_{kr} is the weight on edge (k,r).
2. The procedure to determine if the conditions of Theorem 1 are satisfied is:
   a) Case 1:

      For each edge $(i,j) \notin T$ not incident to node 0, set $\theta_{ij} = w_{ij} - \max(w_i, w_j)$ if $t_i \neq t_j$; otherwise, set θ_{ij} equal to w_{ij} less the maximum weight associated with the arcs on the unique path between nodes i and j. If $\min_{(i,j)} \theta_{ij} \geq 0$, then condition (1) of Theorem 1 is satisfied.

   b) Case 2:

      For each edge $(0,j) \notin T$, set $\theta_{0j} = w_{0j} - w_{0t_j}$. If $\min_{(0,j)} \theta_{0j} \geq 0$, then condition (2) of Theorem 1 is satisfied.

   c) Case 3:

      For each edge $(i,j) \notin T$ not incident to node 0, set $\theta_{ij} = w_{ij} - \max(w_{0t_i}, w_{0t_j})$ and let $\theta = \min_{(i,j)} \theta_{ij}$. For each edge $(0,j) \notin T$, set

$B_{0j} = w_{0j} - w_j$ and let $B = \min_{(0,j)} B_{0j}$. If $\theta + B \geq 0$, then condition (3) of Theorem 1 is satisfied.

If any of the cases of Theorem 1 are not satisfied, improving exchanges can be easily determined via the above evaluation. The labels can be updated after any of these exchanges in the manner discussed in the dual approaches.

4. Order-Constrained One-Trees and Matroid Extensions

By rough analogy to the characterization of a one-tree in [8] we can define an order-constrained one-tree to be a subgraph which as a spanning tree with order at most k at node 0 when node 1 is deleted, and in which node 1 has exactly two incident edges. For k equal to two the minimum order-constrained one-tree problem (defined in the natural manner) is easily established to be a relaxation of the traveling salesman problem. Also an optimal solution to this problem results simply by solving the ordinary minimum spanning tree problem with node 1 deleted, then solving P(2) utilizing the quasi-greedy algorithm of the corollary to theorem 2 if node 0 has an order exceeding two, and finally re-introducing node 1 together with its two incident edges of least weight. Thus the results of this paper provide the basis for a new relaxation strategy for solving the traveling salesman problem. Moreover, as might be expected, these results have direct analogs of greater generality in the context of matroids. These considerations are treated in [7a].

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